Decay Properties of the Connectivity for Mixed Long Range Percolation Models on $\mathbb{Z}^d$

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Abstract

In this short note we consider mixed short-long range independent bond percolation models on $\mathbb{Z}^{k+d}$. Let p_{uv} be the probability that the edge (u, v) will be open. Allowing a x, y-dependent length scale and using a multi-scale analysis due to Aizenman and Newman, we show that the long distance behavior of the connectivity τ_{xy} is governed by the probability p_{xy}. The result holds up to the critical point.

1 Introduction

In this short note we consider a long range percolation model on $\mathbb{L} = (\mathbb{Z}^{k+d}, \mathbb{B})$, where $u \in \mathbb{Z}^{k+d}$ is of the form $u = (\vec{u}_0, \vec{u}_1)$, with $\vec{u}_0 \in \mathbb{Z}^k$ and $\vec{u}_1 \in \mathbb{Z}^d$ and $\mathbb{B}$ is the set of edges (unordered pairs) (u, v), $u \neq v \in \mathbb{Z}^{k+d}$. To each edge (u, v) we associate a Bernoulli random variable ω_b which is open ($\varnothing_b = 1$) with probability

$$p_{uv} = p_{uv}(\beta) \equiv \beta J_{uv}, \qquad u, v \in \mathbb{Z}^{k+d} \tag{1}$$

where $\beta \in [0, 1]$ and, for $\epsilon > 0$, J_{uv} is

$$J_{uv} = \begin{cases} 2(1 + \|\vec{u}_1 - \vec{v}_1\|^{d+\varepsilon})^{-1} & \text{if } \vec{u}_0 = \vec{v}_0 \text{ and } \vec{u}_1 \neq \vec{v}_1; \\ 1 & \text{if } \vec{u}_1 = \vec{v}_1 \text{ and } \|\vec{u}_0 - \vec{v}_0\| = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We denote the event $\{ø \in \emptyset : \text{there is an open path connecting } x \text{ to } y\}$ by $\{x \leftrightarrow y\}$ and define the connectivity function by $\tau_{xy} \equiv P\{x \leftrightarrow y\}$. Let $\|x\| = |x_1| + \cdots + |x_d|$ be the L^1 norm on $\mathbb{Z}^d$ and $\beta_c = \sup\{\beta \in [0,1] : \chi(\beta) < \infty\}$. Our aim is to show the following result

Theorem. *Suppose $\beta < \beta_c$ and consider the long range percolation model with p_{uv} given by (1) and J_{uv} given by (2). Then there exist positive constants $C = C(\beta)$ and $m = m(\beta)$ such that*

$$\tau_{xy} \leq \frac{C \quad e^{-m\|\vec{x}_0 - \vec{y}_0\|}}{1 + \|\vec{x}_1 - \vec{y}_1\|^{d+\varepsilon}} \tag{3}$$

for all $x, y \in \mathbb{Z}^{k+d}$.

The above result says that que probability p_{uv} dictates the long distance behavior of the connectivity function in the subcritical regime (similar lower bounds are easily obtained from the FKG inequality). For the one dimensional $\mathbb{Z}^{0+1}$ percolation model, the above result is known to hold, see [1], the same being true for one dimensional $\mathbb{Z}^{0+1}$ $O(N)$ spin models, $1 \leq N \leq 4$, see [2]. The result is expected to hold in the d-dimensional $\mathbb{Z}^{0+d}$ lattice but it is not clear how to prove it if $\beta < \beta_c$, although one can see it holds if $\beta \approx 0$. Our upper bound (3) holds if $\beta < \beta_c$ and for $(k+d)$-dimensional lattices, $k \geq 0$ and $d \geq 1$. For lattice spin models, the upper bound (3) is known at the high temperature regime, see Ref. [3] for bounded spin models and Ref. [4] for unbounded (and discrete) ones. Ref. [5] extends some of the results of [3, 4] to a general class of continuous spin systems, with J_{uv} given by (1) and $u, v \in \mathbb{Z}^{0+d}$, while [6] considers the more general mixed decay model. In both cases, the polymer expansion (see [7] and references therein) is used and the results hold only in the perturbative regime.

The Hammersley-Simon-Lieb inequality [8, 9, 10] is a key ingredient in [1] and [2] and here we also adopt this "correlation inequality" point of view. For completeness, we state this inequality in the form we will use, see [11]. For each set $S \subset \mathbb{Z}^d$, let $\tau^S_{xy} \equiv P\{x \leftrightarrow y \text{ inside } S\}$. Then

Hammersley-Simon-Lieb Inequality (HSL) *Given $x, y \in \mathbb{Z}^{k+d}$, if $S \subset \mathbb{Z}^{k+d}$ is such that $x \in S$ and $y \in S^c$, then*

$$\tau_{xy} \leq \sum_{\{u \in S, v \in S^c\}} \tau^S_{xu} \, p_{uv} \, \tau_{vy}.$$

We now recall some known facts about the long range percolation model defined by (1) and (2). Let $\theta(\beta, \varepsilon) = P_{\beta,\varepsilon}\{0 \leftrightarrow \infty\}$ be the probability that the origin will be connected to infinity. If $k + d \geq 2$ is the space dimension then, by comparing with the nearest neighbor model and for any positive ε, there exists $\beta_c = \beta_c(d, \varepsilon)$ such that $\theta(\beta, \varepsilon) = 0$ if $\beta < \beta_c$ and $\theta(\beta, \varepsilon) > 0$ if $\beta > \beta_c$. For $k = 0$ and $d = 1$, it is known that the existence of β_c depends upon ε, if $\varepsilon > 1$ then there is no phase transition [12] while it shows up if $0 \leq \varepsilon \leq 1$, see [13]. A phase transition can also be measured in terms of χ, the mean cluster size, given by $\chi = \sum_x \tau_{0x}$. Let $\pi_c(d,\varepsilon) = \sup\{\beta : \chi(\beta,\varepsilon) < \infty\}$. Then, it comes from the FKG Inequality [14] that $\pi_c(d,\varepsilon) \leq \beta_c(d,\varepsilon)$. The equality $\pi_c(d,\varepsilon) = \beta_c(d,\varepsilon)$ holds for the class of models we are dealing with and it was proved independently by Aizenman and Barsky in [15] and Menshikov in [16]. We will use the condition $\chi < \infty$ to characterize the subcritical region.

The remaining of this note is divided as follows: in the next section we prove Theorem 1 and in Section 3 we make some concluding remarks regarding the validity of our results to ferromagnetic spin models.

2 Proof of the Theorem

Let $x = (\vec{x}_0, \vec{x}_1) \in \mathbb{Z}^{k+d}$. We first observe that

$$\chi = \sum_{n \geq 0} \sum_{\|\vec{x}_0\| = n} \sum_{\vec{x}_1 \in \mathbb{Z}^d} \tau_{0x}.$$

Since $\chi < \infty$, given $l \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have

$$\sum_{\|\vec{x}_0\| = n} \sum_{\vec{x}_1 \in \mathbb{Z}^d} \tau_{0x} < l.$$

Consider now $x = (\vec{x}_0, \vec{x}_1)$ with $\|\vec{x}_0\| > n_0$. Using the translation invariance of the model and applying iteratively the HSL Inequality with $S = \{x \in \mathbb{Z}^{k+d}; \|\vec{x}_0\| \leq n_0\}$, we obtain

$$\sum_{\vec{x}_1 \in \mathbb{Z}^d} \tau_{0x} \leq l^{\lfloor \|\vec{x}_0\|/n_o \rfloor} \leq C_1 \exp(-(m+\delta)\|\vec{x}_0\|),$$

where $\lfloor r \rfloor$ denotes the integer part of r, $\delta > 0$ is given and m is defined by $e^{-(m+\delta)} = \lambda^{1/n_0}$. Next we show that a HSL type inequality holds for the modified connectivity function $T_m(x,y) \equiv e^{m\|\vec{x}_0 - \vec{y}_0\|}\tau_{xy}$ and for the set $S = \mathcal{C}_r(x) \equiv \{z \in \mathbb{Z}^{k+d}; \|\vec{x}_1 - \vec{z}_1\| \leq r\}$. Applying the HSL Inequality with the above specified S, we have

$$\tau_{xy} \leq \sum_{\substack{u \in \mathcal{C}_r(x) \\ v \in \mathcal{C}_r^c(x)}} \tau_{xu} p_{uv} \tau_{vy}.$$

Then, for $y \in \mathcal{C}_L^c(x)$ for some $L > 1$, we obtain

$$\begin{aligned} T_m(x,y) &= e^{m\|\vec{x}_0 - \vec{y}_0\|}\tau_{xy} \leq e^{m\|\vec{x}_0 - \vec{y}_0\|} \sum_{\substack{u \in \mathcal{C}_L(x) \\ v \in \mathcal{C}_L^c(x)}} \tau_{xu} p_{uv} \tau_{vy} \\ &\leq \sum_{\substack{u \in \mathcal{C}_L(x) \\ v \in \mathcal{C}_L^c(x)}} e^{m\|\vec{x}_0 - \vec{u}_0\|}\tau_{xu} \quad p_{uv} \quad e^{m\|\vec{y}_0 - \vec{v}_0\|}\tau_{vy} \end{aligned}$$

since we necessarily have that $\vec{u}_0 = \vec{v}_0$. It then follows that

$$T_m(x,y) \le \sum_{\substack{u\in\mathcal{C}_L(x)\\ v\in\mathcal{C}^c_L(x)}} T_m(x,u)p_{uv}T_m(v,y).$$

We remark that $\chi_m \equiv \sum_{x\in\mathbb{Z}^{k+d}} T_m(0,x) < \infty$ is finite if $\beta < \beta_c$ since

$$\begin{aligned}\sum_{x\in\mathbb{Z}^{k+d}} T_m(0,x) = \sum_{x\in\mathbb{Z}^{k+d}} e^{m\|\vec{x}_0\|}\tau_{0x} &\le \sum_{k\ge 0}\sum_{\|\vec{x}_0\|=k} e^{m\|\vec{x}_0\|}\sum_{\vec{x}_1\in\mathbb{Z}^d}\tau_{0x}\\ &\le \sum_{k\ge 0} 2dk^{d-1}e^{-\delta k} < \infty.\end{aligned}$$

From now on we closely follow Section 3 of [1] and prove the polynomial decay of T_m up to the critical point. For fixed $x, y \in \mathbb{Z}^{k+d}$ and $L \equiv \|\vec{x}_1 - \vec{y}_1\|/4$, we know that

$$\begin{aligned} T_m(x,y) &\le \sum_{\substack{u\in\mathcal{C}_L(x)\\ v\in\mathcal{C}^c_L(x)}} T_m(x,u)p_{uv}T_m(v,y)\\ &\le \sum_{\substack{u\in\mathcal{C}_L(x)\\ v\in\mathcal{C}^c_L(x)\cap\mathcal{C}_{3L}(x)}} T_m(u,x)p_{uv}T_m(v,y) + \sum_{\substack{u\in\mathcal{C}_L(x)\\ v\in\mathcal{C}^c_L(x)\cap\mathcal{C}^c_{3L}(x)}} T_m(x,u)p_{uv}T_m(v,y). \end{aligned} \tag{4}$$

Let

$$\mathbb{T}_m(L) \equiv \sup\{T_m(0,u); u\in\mathcal{C}^c_L(0)\} \quad \text{and} \quad \gamma_L \equiv \sum_{\substack{u\in\mathcal{C}_L(x)\\ v\in\mathcal{C}^c_L(x)}} T_m(x,u)p_{uv}.$$

Then, the first term on the r.h.s. of (4) is bounded above by $\mathbb{T}_m(L/2)\gamma_L$ while the second one is bounded by

$$\frac{2^{d+\varepsilon}2\beta\chi_m^2}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}},$$

leading to

$$T_m(x,y) \le \frac{2^{d+\varepsilon}2\beta\chi_m^2}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}} + \gamma_L\mathbb{T}_m\left(\frac{L}{2}\right).$$

Now, since $\chi_m < \infty$ for $\beta < \beta_c$ and since $\sum_u p_{0u} < \infty$, we have that $\gamma_L \to 0$ as $L \to \infty$. For $\alpha \in (0, 2^{-(d+\varepsilon)})$, there exists $L_0 > 0$ such that $\gamma_L < \alpha$ for all $L \ge L_0$. Considering $L > L_0$, it follows that

$$\mathbb{T}_m(L) \le \frac{2^{d+\varepsilon}2\beta\chi_m^2}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}} + \alpha\mathbb{T}_m\Big(\frac{L}{2}\Big). \tag{5}$$

Iterating (5) n times, with n the smallest integer for which $L2^{-n} \le L_0$, we have for all $L > L_0$

$$\mathbb{T}_m(L) \le \frac{2\beta\chi_m^2\sum_{j=0}^{n-1}(\alpha 2^{d+\varepsilon})^j}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}} + \alpha^n\mathbb{T}_m\Big(\frac{L}{2^n}\Big).$$

Noting that $T_m(x,y) \le \mathbb{T}_m(L)$, that $\mathbb{T}_m(L) \le 1$ for any $L > 0$ and that

$$\alpha^n \le 2^{-(d+\varepsilon)n} = \frac{1}{(1+L^{d+\varepsilon})}\frac{(1+L^{d+\varepsilon})}{2^{(d+\varepsilon)n}} \le \frac{2\cdot 2^{d+\varepsilon}}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}}\left(\frac{L}{2^n}\right)^{d+\varepsilon} \le \frac{2(2L_0)^{d+\varepsilon}}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}},$$

we can conclude that, for $\beta < \beta_c$,

$$\tau_{xy} \le \frac{C \quad e^{-m\|\vec{x}_0-\vec{y}_0\|}}{1+\|\vec{x}_1-\vec{y}_1\|^{d+\varepsilon}}$$

and the bound (3) holds. □

3 Concluding Remarks

The strategy used to prove the main Theorem can also be applied to $\mathbb{Z}^d$ ferromagnetic models with free boundary conditions and pair interaction J_{uv} given by (2). For this class of models the Griffiths inequalities [17, 18] are valid, guaranteeing the positivity of spin-spin correlations $\langle\sigma_x\sigma_y\rangle$. Simon-Lieb Inequality also holds in this case (see [7] and references therein), with p_{uv} replaced by βJ_{uv}, where β is the inverse of the temperature, and with τ_{xy} replaced by $\langle\sigma_x\sigma_y\rangle$. Finally, since the uniqueness of the critical point is guaranteed in [19], the results of Section 2 are also valid for these models.

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